Limits - Vol.1

Q1. $\lim\limits_{x \to \infty} x \sin \frac{1}{x} = ?$

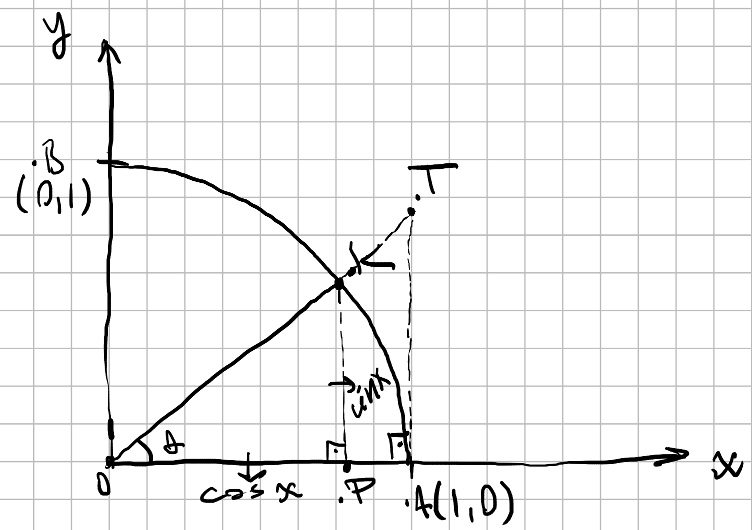
$$\lim_{x \to \infty} x \sin\frac{1}{x} = \lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \Bigg\} \quad \frac{1}{x} = \theta$$

if $\lim\limits_{x \to \infty}$ then, $\lim\limits_{\theta \to 0}$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Proof of $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

So, when we first just write the x values we face with a $\frac{0}{0}$ uncertainty. We gotta fix that. Let's create a $\frac{1}{4}$ unit circle.



We should use the "Squeeze Theorem" which is also known as "Sandwich Theorem".

A(blabla) ≡ Area of blabla.

$$A(\overset{\triangle}{OPK}) \leq \frac{\theta}{2} \leq A(\overset{\triangle}{OAT})$$

To clarify "$\frac{\theta}{2}$" here,

A() $= A\left(\frac{\theta}{2\pi} \cdot \pi \cdot r^2\right)$ or $A\left(\frac{\theta}{180°} \cdot \pi \cdot r^2\right)$,

where $r = 1$.

$A(O\hat{P}K) \leq \frac{\theta}{2} \leq A(O\hat{A}T) \Leftrightarrow \frac{\sin\theta \cdot \cos\theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan\theta}{2}$

Let's save these terms from their denominators.

$\sin\theta \cdot \cos\theta \leq \theta \leq \tan\theta$, and now lets rewrite all the terms over $\sin\theta$.

$\cos\theta \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$

So, when we use the squeeze theorem,

$\lim_{\theta \to 0} \left(\cos\theta \leq \frac{\sin\theta}{\theta} \leq \frac{1}{\cos\theta} \right)$

So, $\cos 0 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq \frac{1}{\cos 0}$. $\cos 0 = 1 \Leftrightarrow \frac{1}{\cos 0} = 1$

So, $1 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Q2. Show that $\lim_{x \to 3} 4 = 4$ by using Epsilon - Delta proof.

Before solving the question let's start with the definitions of Epsilon (ε) and Delta (δ).

ε - Epsilon's Definition

for $\lim_{x \to a} f(x) = L$, meaning ε: how close $f(x)$ can be to L. ($\varepsilon > 0$)

So what we are looking for is the max. value of $|f(x) - L|$. So, $\varepsilon > |f(x) - L|$.

δ - Delta's Definition

just like Epsilon, $\delta > 0$. Meaning δ: how close x can be to a.

So, $\delta > |x - a| > 0$.

So, now let's solve the questio

Question again| $\delta-\varepsilon$ proof of $\lim_{x\to 3} 4=4$? // (50 points)

Solution//

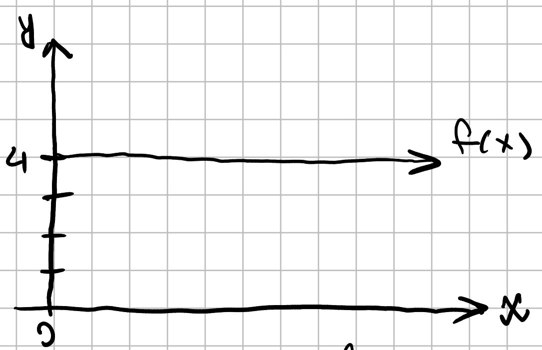
pf:

$\lim_{x\to 3} 4=4$, so

We say $\lim_{x\to a} f(x)=L$.

if $\forall\varepsilon>0, \exists\delta>0$ such that,

$0<|x-a|<\delta$ and $\varepsilon>|f(x)-L|$.



so, for this proof:

$a=3$
$f(x)=4$
$L=4$

$|f(x)-L|<\varepsilon$, $|4-4|<\varepsilon$

so $\varepsilon>0$.

This statament is allways true for any value of δ.

Since $|0|<\varepsilon$,

$|0(x-3)|<\varepsilon$.

$0\cdot|x-3|<\varepsilon$.

So, Delta(δ) is infinitely Large.